

Exponentiating 2×2 and 3×3 Matrices Done Right

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Abstract

We derive explicit formulas for calculating e^A , $\cosh A$, $\sinh A$, $\cos A$ and $\sin A$ for a given 2×2 matrix A . We also derive explicit formulas for e^A for a given 3×3 matrix A . These formulas are expressed exclusively in terms of the characteristic roots of A and involve neither the eigenvectors of A , nor the transition matrix associated with a particular canonical basis. We believe that our method has advantages (especially if applied by non-mathematicians or students) over the more conventional methods based on the choice of canonical bases. We support this point with several examples for solving first order linear systems of ordinary differential equations with constant coefficients.

Key words: Exponential of a matrix, characteristic polynomial, Cayley-Hamilton theorem, nilpotent matrix, projection, transition matrix, linear system of ordinary differential equations.

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1 Introduction

The exponential e^A of a square matrix A and the related one-parameter family e^{tA} are important concepts in mathematics. Here is one example (among many others): Let $\vec{x}'(t) = A\vec{x}(t)$ be a system of first order homogeneous ordinary differential equations with constant coefficients with initial conditions $\vec{x}(0) = \vec{x}_0 \in \mathbb{R}^n$, where A is an $n \times n$ matrix with real entries. Then the solution to the system is given by the formula $\vec{x}(t) = e^{tA} \vec{x}_0$ (Michael Artin [1], p. 140).

The exponential e^{tA} is defined by the Taylor expansion: $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. For some particular matrices A the exponential can be easily calculated. For example, if N is a nilpotent matrix of order m , i.e. $N^m = O$, then $e^{tN} = \sum_{n=0}^{m-1} \frac{t^n N^n}{n!}$. If P is a projection, i.e. $P^2 = P$, then $e^{tP} = I + \sum_{n=1}^{\infty} \frac{t^n P}{n!} = I - P + e^t P$. If $A = \alpha I$ for some scalar α , then $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n \alpha^n}{n!} I = e^{\alpha t} I$. From the Taylor expansion it follows that if $AB = BA$ for two square matrices, then $e^{A+B} = e^A e^B$. Also, if A and C are similar matrices with a transition matrix T , i.e. $A = TCT^{-1}$, then $e^A = Te^C T^{-1}$.

These facts can be found in many textbooks (Jerry Farlow et al [2], p. 350) and handbooks ([5], p. 132-133) on linear algebra. For a general matrix A , however, the rule for calculating e^{tA} becomes somewhat more complicated (A.I. Malcev [3], p. 118 or Michael Artin [1], p. 480-482): We have to find the Jordan form $T^{-1}AT = \text{diag}(C_1, C_2, \dots, C_k)$ of the matrix A , where C_i 's are Jordan cells. Thus the formula $e^A = Te^C T^{-1}$ becomes $e^A = T\text{diag}(e^{C_1}, e^{C_2}, \dots, e^{C_k})T^{-1}$. On the other hand, each cell can be presented in the form $C_i = \lambda_i I + N$, where $N^{m_i} = O$, λ_i is the corresponding eigenvalue of A and $m_i = \text{size}(C_i)$. As a result, $e^{C_i} = e^{\lambda_i} \sum_{n=0}^{m_i-1} \frac{N^n}{n!}$. The procedure is universal and it looks attractive but it has the following disadvantages: (a) The algorithm for calculating the transition matrix T and its inverse T^{-1} is difficult and time consuming. It is especially complicated in the case of multiple characteristic roots, when the calculations of the generalized eigenvectors require skills that are too advanced for students in a typical course on ordinary differential equations and linear algebra. (b) The original framework \mathbb{R}^n must be extended to \mathbb{C}^n , which again might be confusing for students. Our observation is that students and non-mathematicians (the latter might have rusty knowledge of linear algebra) who rarely calculate exponentials of matrices often have considerable difficulty calculating e^{tA} even for 2×2 and 3×3 matrices.

In this article we derive explicit formulas for calculating e^{tA} for a 2×2

and 3×3 matrix A with real entries. In the case of 2×2 matrices A we also derive explicit formulas for $\cosh A$, $\sinh A$, $\cos A$ and $\sin A$. These formulas are expressed exclusively in terms of the characteristic roots of A and involve neither the eigenvectors of A , nor the transition matrix associated with a particular canonical basis. We believe that our formulas are suitable for handbooks in the sense that one can apply them with limited background in linear algebra, in particular, without having slightest idea what a “transition matrix is”. The formulas are derived with the help the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I)$ of A and the Cayley-Hamilton theorem which says that $f_A(A) = 0$. But, again, the formulas can be applied without knowledge of the Cayley-Hamilton theorem. In Corollary 2.1 and Corollary 4.1 we identify some matrices closely related to A which allow easy calculation of the invariant subspaces of A and the canonical bases (if needed). We recommend our method for teaching exponentials in a course on linear algebra and ordinary differential equations. To support this point we present several examples for solving linear systems of ordinary differential equations.

For other methods for calculating the exponential of a matrix not mentioned in our paper we refer to the survey article (Cleve Moler and Charles Van Loan [4]), where the reader will find more references to the subject.

In what follows I and O denote the identity and zero square matrices, respectively. We denote by $[\vec{x} \mid \vec{y} \mid \vec{z} \mid \dots]$ the matrix with column-vectors $\vec{x}, \vec{y}, \vec{z}, \dots \in \mathbb{R}^n$. Also, $E_{\lambda=\mu}(A)$ denotes the eigenspace of A corresponding to $\mu \in \mathbb{C}$.

2 Exponential of a 2×2 Matrix

We derive formulas for calculating the exponential of a given 2×2 -matrix A based exclusively on the characteristic roots of A . Our formulas are easy to memorize and simple to use in the sense that: (a) our framework is always \mathbb{R}^2 (never \mathbb{C}^2); (b) our formulas involve linear operations between matrices only; (c) our formulas can be handled with limited background in linear algebra - they involve neither the eigenvectors of A , nor a particular canonical basis (although we need the characteristic roots of A). We illustrate this point with several examples of linear systems of ordinary differential equations. Although we do not need a change of the basis to calculate the exponential e^{tA} , we also derive explicit formulas for a canonical basis and a transition matrix T for which $C = T^{-1}AT$ is exceptionally simple for the purpose of

the exponentiation (Corollary 2.1).

Theorem 2.1 *Let A be a 2×2 -matrix with real entries and let λ_1 and λ_2 be the characteristic roots of A .*

Case 1: *If $\lambda_1 = \lambda_2$ (real), then $A = \lambda_0 I + N$, where $\lambda_0 = \lambda_1 = \lambda_2$ and the matrix $N = A - \lambda_0 I$ satisfies the equation $N^2 = O$. Consequently, for every real (or complex) t we have*

$$(1) \quad e^{tA} = e^{\lambda_0 t}(I + tN).$$

Case 2: *If $\lambda_{1,2} = \alpha \pm i\omega$ for some $\alpha, \omega \in \mathbb{R}$, $\omega \neq 0$, then $A = \alpha I + \omega J$, where the matrix $J = \frac{1}{\omega}(A - \alpha I)$ satisfies the equation $J^2 = -I$. Consequently, for every real (or complex) t we have*

$$(2) \quad e^{tA} = e^{\alpha t}[(\cos \omega t)I + (\sin \omega t)J].$$

Case 3: *If $\lambda_1 \neq \lambda_2$ (both real), then $A = \alpha I + \beta J$, where $\alpha = \frac{\lambda_1 + \lambda_2}{2}$, $\beta = \frac{\lambda_1 - \lambda_2}{2}$ and the matrix $J = \frac{1}{\beta}(A - \alpha I)$ satisfies the equation $J^2 = I$. Consequently, for every real (or complex) t we have*

$$(3) \quad e^{tA} = e^{\alpha t}\{(\cosh \beta t)I + (\sinh \beta t)J\} \stackrel{\text{or}}{=} \frac{1}{2}[e^{\lambda_1 t}(I + J) + e^{\lambda_2 t}(I - J)].$$

Proof: Case 1: We first show that $N^2 = 0$. Indeed, $N^2 = (A - \lambda_0 I)^2 = A^2 - 2\lambda_0 A + \lambda_0^2 I = O$, by the Cayley-Hamilton theorem, since $2\lambda_0 = \text{tr}(A)$ and $\lambda_0^2 = \det(A)$. Thus we have

$$e^{tA} = e^{t(\lambda_0 I + N)} = e^{\lambda_0 tI} \cdot e^{tN} = e^{\lambda_0 t} I (I + tN + 0 + \dots) = e^{\lambda_0 t} (I + tN),$$

as required.

Case 2: We first show that $J^2 = -I$. Indeed, we have $A^2 - 2\alpha I + (\alpha^2 + \omega^2)I = O$, by the Cayley-Hamilton theorem, since $2\alpha = \lambda_1 + \lambda_2 = \text{tr}(A)$ and $\alpha^2 + \omega^2 = \lambda_1 \lambda_2 = \det(A)$. Thus $J^2 = (\frac{1}{\omega}(A - \alpha I))^2 = \frac{1}{\omega^2}(A^2 - 2\alpha I + \alpha^2 I) = \frac{1}{\omega^2}(A^2 - 2\gamma I + (\alpha^2 + \omega^2)I - \omega^2 I) = \frac{1}{\omega^2}(O - \omega^2 I) = -I$. Next, we calculate

$$\begin{aligned} e^{tA} &= e^{t(\alpha I + \omega J)} = e^{\alpha tI} e^{\omega tJ} = e^{\alpha t} \left\{ \sum_{n=0}^{\infty} \frac{(\omega t)^{2n} J^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\omega t)^{2n+1} J^{2n+1}}{(2n+1)!} \right\} = \\ &= e^{\alpha t} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n (\omega t)^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(-1)^n (\omega t)^{2n+1}}{(2n+1)!} J \right\} = e^{\alpha t} \{(\cos \omega t)I + (\sin \omega t)J\}, \end{aligned}$$

as required. \blacktriangleleft

Case 3: We shall, first, show that $J^2 = I$. Indeed, we have $A^2 - 2\alpha I + (\alpha^2 - \beta^2)I = O$, by the Cayley-Hamilton theorem, since $2\alpha = \lambda_1 + \lambda_2 = \text{tr}(A)$ and $\alpha^2 - \beta^2 = \lambda_1 \lambda_2 = \det(A)$. Thus $J^2 = (\frac{1}{\beta}(A - \alpha I))^2 = \frac{1}{\beta^2}(A^2 - 2\alpha I + \alpha^2 I) = \frac{1}{\beta^2}(A^2 - 2\alpha I + (\alpha^2 - \beta^2)I + \beta^2 I) = \frac{1}{\beta^2}(0 + \beta^2 I) = I$. Next, we calculate

$$\begin{aligned} e^{tA} &= e^{t(\alpha I + \beta J)} = e^{\alpha t I} \cdot e^{\beta t J} = e^{\alpha t} \left\{ \sum_{n=0}^{\infty} \frac{(\beta t)^{2n} J^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\beta t)^{2n+1} J^{2n+1}}{(2n+1)!} \right\} = \\ &= e^{\alpha t} \left\{ \sum_{n=0}^{\infty} \frac{(\beta t)^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(\beta t)^{2n+1}}{(2n+1)!} J \right\} = e^{\alpha t} \{(\cosh \beta t)I + (\sinh \beta t)J\}, \end{aligned}$$

as required. \blacktriangleleft

The formulas (1)-(3) show that we do not need the eigenvectors of A in order to calculate e^{tA} . However, the eigenvectors of A as well as canonical forms and canonical bases (if needed for classification, graphing, etc.) can be easily extracted from the matrices N and J . The next corollary follows easily from the above theorem and we leave the proof to the reader.

Corollary 2.1 (Eigenvectors, Canonical Forms and Bases) *Under the notation of the previous theorem we have the following:*

Case 1: If $N \neq O$, then $E_{\lambda=\lambda_0}(A) = \text{Im}(N)$. In particular, either of the non-zero columns of N is an eigenvector of A . We have $A = T \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} T^{-1}$, where $T = [N\vec{x} | \vec{x}]$ and $\vec{x} \in \mathbb{R}^2$ such that $N\vec{x} \neq \vec{0}$. If $N = O$, then $A = \lambda_0 I$.

Case 2: The matrix A does not have eigenvectors in \mathbb{R}^2 . We have $A = T \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} T^{-1}$, where $T = [J\vec{x} | \vec{x}]$ and $\vec{x} \in \mathbb{R}^2, \vec{x} \neq \vec{0}$. Notice that the matrix $\begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}$ is conformal (i.e. the corresponding transformation preserves the angles in \mathbb{R}^2). Alternatively, in \mathbb{C}^2 the matrix A is diagonalizable with eigenvalues $\alpha \pm i\omega$ and eigenvectors $(I \mp iJ)\vec{x}, \vec{x} \neq \vec{0}$, respectively.

Case 3: We have $E_{\lambda=\lambda_{1,2}}(A) = \text{Im}(I \pm J)$, respectively. We have $A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$, where $T = [(I+J)\vec{x} | (I-J)\vec{y}]$ with $\vec{x}, \vec{y} \in \mathbb{R}^2$ such that $(I+J)\vec{x} \neq \vec{0}$ and $(I-J)\vec{y} \neq \vec{0}$.

Here are several examples tested in class by the second author in a course on linear algebra and ordinary differential equations. We strongly recommend

these formulas for teaching exponentials.

Example 2.1 *We shall find the solution of the initial value problem without involving eigenvectors:*

$$\begin{aligned} x' &= 3x + 2y, \\ y' &= -8x - 5y, \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = -1$. We have $A = \begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix}$ and $\vec{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 + 2\lambda + 1$ with roots $\lambda_1 = \lambda_2 = -1$ (Case 1). We calculate

$$N = A + I = \begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix}.$$

We leave to the reader to verify that $N^2 = O$. For the exponential of A we apply formula (1):

$$\vec{x}(t) = e^{tA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} 1+4t & 2t \\ -8t & 1-4t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} 1+2t \\ -1-4t \end{pmatrix}.$$

Thus $x = e^{-t}(1+2t)$, $y = -e^{-t}(1+4t)$.

Remark 2.1 *If a canonical form for A and a transition matrix are still needed, we can easily calculate them with the help of Corollary 2.1. For example, $T^{-1}AT = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, where $T = [N\vec{x} | \vec{x}] = \begin{pmatrix} 2 & 0 \\ -4 & 1 \end{pmatrix}$ for $\vec{x} = (0, 1)$. Notice that either of the columns of N is an eigenvector of A .*

Example 2.2 *We shall find the solution to the initial value problem without involving eigenvectors:*

$$\begin{aligned} x' &= y, \\ y' &= -5x - 2y, \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = 1$. We have $A = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$ and $\vec{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 + 2\lambda + 5$ with roots $\lambda_{1,2} =$

$-1 \pm 2i$, thus $\alpha = -1$ and $\omega = 2$ (Case 2). We calculate

$$J = \frac{1}{\omega}(A - \alpha I) = \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}.$$

We leave to the reader to verify that $J^2 = -I$. For the solution to the system we apply formula (2):

$$\begin{aligned} \vec{x} &= e^{tA}\vec{x}_0 = e^{-t} \left[(\cos 2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\sin 2t) \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \\ &= e^{-t} \begin{pmatrix} \cos 2t + \frac{1}{2} \sin 2t, & \frac{1}{2} \sin 2t \\ -\frac{5}{2} \sin 2t, & \cos 2t - \frac{1}{2} \sin 2t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = e^{-t} \begin{pmatrix} 2 \cos 2t + \frac{3}{2} \sin 2t \\ \cos 2t - \frac{11}{2} \sin 2t \end{pmatrix} \end{aligned}$$

Thus $x = e^{-t}(2 \cos 2t + \frac{3}{2} \sin 2t)$, $y = e^{-t}(\cos 2t - \frac{11}{2} \sin 2t)$.

Remark 2.2 If a canonical form for A and the corresponding transition matrix are still needed, we can easily calculate them with the help of Corollary 2.1. For example, $T^{-1}AT = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$, where $T = [J\vec{x} | \vec{x}] = \begin{pmatrix} 1 & 2 \\ -5 & 0 \end{pmatrix}$ for $\vec{x} = (2, 0)$. In the framework of \mathbb{C}^2 we have $T^{-1}AT = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}$, where $T = [(I-iJ)\vec{x} | (I+iJ)\vec{x}] = \begin{pmatrix} 2+i & 2-i \\ -5i & 5i \end{pmatrix}$ for $\vec{x} = (2, 0)$.

Example 2.3 We shall find the solution to the initial value problem without involving eigenvectors:

$$\begin{aligned} x' &= 5x - y, \\ y' &= 3x + y, \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = 2$. We have $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$ and $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 - 6\lambda + 8$ with roots $\lambda_1 = 4$, $\lambda_2 = 2$ (Case 3). We calculate $\alpha = 3$, $\beta = 1$ and

$$J = \frac{1}{\beta}(A - \alpha I) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$

We leave to the reader to verify that $J^2 = I$. For the solution to the system we apply formula (3):

$$\begin{aligned}\vec{x}(t) &= e^{tA} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{3t} \begin{pmatrix} \cosh t + 2 \sinh t, & -\sinh t \\ 3 \sinh t, & \cosh t - 2 \sinh t \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \\ &= e^{3t} \begin{pmatrix} \cosh t \\ 2 \cosh t - \sinh t \end{pmatrix}.\end{aligned}$$

Thus $x = e^{3t} \cosh t \stackrel{\text{or}}{=} \frac{1}{2}(e^{4t} + e^{2t})$ and $y = e^{3t}(2 \cosh t - \sinh t) \stackrel{\text{or}}{=} \frac{1}{2}(e^{4t} + 3e^{2t})$.

Remark 2.3 If a canonical form for A and the transition matrix are still needed, we can easily calculate them with the help of Corollary 2.1. For example, $T^{-1}AT = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, where $T = [(I+J)\vec{x} \mid (I-J)\vec{x}] = \begin{pmatrix} -1 & 1 \\ -1 & 3 \end{pmatrix}$ for $\vec{x} = (0, 1)$. Notice that the columns of T are eigenvectors of A .

3 Some Applications

If A is a square matrix, we define the matrices $\cosh A$, $\sinh iA$, e^{iA} , $\cos A$ and $\sin A$ by the corresponding Taylor series. For example, $\cosh A = \sum_{n=0}^{\infty} \frac{A^{2n}}{(2n)!}$, and the rest are defined similarly. In this section we derive explicit formulas for these elementary functions which follow easily from Theorem 2.1.

Corollary 3.1 (Hyperbolic Functions) Let A be a 2×2 -matrix with real entries and let λ_1 and λ_2 be the characteristic roots of A .

Case 1: If $\lambda_1 = \lambda_2$ (real), then

$$\cosh A = (\cosh \lambda_0)I + (\sinh \lambda_0)N, \quad \sinh A = (\sinh \lambda_0)I + (\cosh \lambda_0)N,$$

where $\lambda_0 = \lambda_1 = \lambda_2$ and $N = A - \lambda_0 I$.

Case 2: If $\lambda_{1,2} = \alpha \pm i\omega$ for some $\alpha, \omega \in \mathbb{R}$, $\omega \neq 0$, then

$$\begin{aligned}\cosh A &= (\cosh \alpha)(\cos \omega)I + (\sinh \alpha)(\sin \omega)J, \\ \sinh A &= (\sinh \alpha)(\cos \omega)I + (\cosh \alpha)(\sin \omega)J,\end{aligned}$$

where $J = \frac{1}{\omega}(A - \alpha I)$.

Case 3: If $\lambda_1 \neq \lambda_2$ (both real), then

$$\begin{aligned}\cosh A &= (\cosh \alpha)(\cosh \beta)I + (\sinh \alpha)(\sinh \beta)J, \\ \sinh A &= (\sinh \alpha)(\cosh \beta)I + (\cosh \alpha)(\sinh \beta)J,\end{aligned}$$

where $\alpha = \frac{\lambda_1 + \lambda_2}{2}$, $\beta = \frac{\lambda_1 - \lambda_2}{2}$ and $J = \frac{1}{\beta}(A - \alpha I)$.

Proof: The results follow directly from Theorem 2.1 and the formulas $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$. \blacktriangle

Corollary 3.2 (Complex Exponent) *Let A be a 2×2 -matrix with real entries and let λ_1 and λ_2 be the characteristic roots of A .*

Case 1: *If $\lambda_1 = \lambda_2$ (real), then*

$$e^{\pm iA} = e^{\pm i\lambda_0}(I \pm iN),$$

where $\lambda_0 = \lambda_1 = \lambda_2$ and $N = A - \lambda_0 I$.

Case 2: *If $\lambda_{1,2} = \alpha \pm i\omega$ for some $\alpha, \omega \in \mathbb{R}$, $\omega \neq 0$, then*

$$e^{\pm iA} = e^{\pm i\alpha}[(\cosh \omega)I \pm i(\sinh \omega)J],$$

where $J = \frac{1}{\omega}(A - \alpha I)$.

Case 3: *If $\lambda_1 \neq \lambda_2$ (both real), then*

$$e^{\pm iA} = e^{\pm i\alpha}\{(\cos \beta)I \pm i(\sin \beta)J\},$$

where $\alpha = \frac{\lambda_1 + \lambda_2}{2}$, $\beta = \frac{\lambda_1 - \lambda_2}{2}$ and $J = \frac{1}{\beta}(A - \alpha I)$.

Proof: These formulas follow directly from Theorem 2.1 for $t = i$. \blacktriangle

Corollary 3.3 (Trigonometric Functions) *Let A be a 2×2 -matrix with real entries and let λ_1 and λ_2 be the characteristic roots of A .*

Case 1: *If $\lambda_1 = \lambda_2$ (real), then*

$$\cos A = (\cos \lambda_0)I - (\sin \lambda_0)N, \quad \sin A = (\sin \lambda_0)I + (\cos \lambda_0)N,$$

where $\lambda_0 = \lambda_1 = \lambda_2$ and $N = A - \lambda_0 I$.

Case 2: *If $\lambda_{1,2} = \alpha \pm i\omega$ for some $\alpha, \omega \in \mathbb{R}$, $\omega \neq 0$, then*

$$\begin{aligned} \cos A &= (\cos \alpha)(\cosh \omega)I - (\sin \alpha)(\sinh \omega)J, \\ \sin A &= (\sin \alpha)(\cosh \omega)I + (\cos \alpha)(\sinh \omega)J, \end{aligned}$$

where $J = \frac{1}{\omega}(A - \alpha I)$.

Case 3: *If $\lambda_1 \neq \lambda_2$ (both real), then*

$$\begin{aligned} \cos A &= (\cos \alpha)(\cos \beta)I - (\sin \alpha)(\sin \beta)J, \\ \sin A &= (\sin \alpha)(\cos \beta)I + (\cos \alpha)(\sin \beta)J. \end{aligned}$$

where $\alpha = \frac{\lambda_1 + \lambda_2}{2}$, $\beta = \frac{\lambda_1 - \lambda_2}{2}$ and $J = \frac{1}{\beta}(A - \alpha I)$.

Proof: As before, the result follows directly from Corollary 3.2 and the formulas $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$. \blacktriangle

4 Exponential of a 3×3 Matrix

In this section we extend the result about e^A , presented in Section 2, for 3×3 matrices A . We should notice that: (a) the hardest (and the most time consuming) parts of our formulas for e^A are the calculating the squares of some matrices; we need neither to solve linear systems of equations, nor to invert matrices: (b) our framework is always \mathbb{R}^3 (never \mathbb{C}^3). For that reason we believe that our method has advantages over the most conventional methods based on the choice of a particular canonical basis with transition matrix T and calculating its inverse T^{-1} . To support this point we present several examples for solving linear systems of ordinary differential equations in three unknowns.

In what follows we shall use the fact that if A is a matrix and $B = A - aI$ for some real a , then $f_B(\lambda) = f_A(\lambda + a)$.

Theorem 4.1 *Let A be a 3×3 matrix with real entries and let λ_1, λ_2 and λ_3 be the characteristic roots of A . Then:*

Case 1: *If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_0$, then $N = A - \lambda_0 I$ is a nilpotent matrix such that $N^3 = 0$. Consequently, for every $t \in \mathbb{R}$ we have*

$$(4) \quad e^{tA} = e^{\lambda_0 t} \left(I + tN + \frac{t^2}{2} N^2 \right).$$

Case 2: *If $\lambda_1 = \lambda_2 \neq \lambda_3$, then $A = \lambda_0 I + (\lambda_3 - \lambda_0)P + N$, where $\lambda_0 = \lambda_1 = \lambda_2$,*

$$P = \frac{1}{(\lambda_3 - \lambda_0)^2} (A - \lambda_0 I)^2 \text{ and } N = A - \lambda_0 I - (\lambda_3 - \lambda_0)P.$$

Also, we have $PN = NP = 0$, $P^2 = P$ and $N^2 = 0$. Consequently, for every $t \in \mathbb{R}$ we have

$$(5) \quad e^{tA} = e^{\lambda_0 t} (I - P + tN) + e^{\lambda_3 t} P.$$

Case 3: *Let $\lambda_{1,2} = \alpha \pm i\omega$ for some $\alpha, \omega \in \mathbb{R}$, $\omega \neq 0$. Then $A = \alpha I + (\lambda_3 - \alpha)P + \omega J$, where*

$$P = \frac{(A - \alpha I)^2 + \omega^2 I}{(\lambda_3 - \alpha)^2 + \omega^2}, \quad J = \frac{1}{\omega} [A - \alpha I - (\lambda_3 - \alpha)P].$$

Also, we have $PJ = JP = 0$, $P^2 = P$, $J^{2n} = (-1)^n(I - P)$, $n = 1, 2, \dots$, $J^{2n+1} = (-1)^nJ$, $n = 0, 1, 2, \dots$. Consequently, for every $t \in \mathbb{R}$ we have

$$(6) \quad e^{tA} = e^{\alpha t} [(\cos \omega t)(I - P) + (\sin \omega t)J] + e^{\lambda_3 t}P,$$

Case 4: If $\lambda_1, \lambda_2, \lambda_3$ are three distinct reals, then $A = \alpha I + (\lambda_3 - \alpha)P + \beta J$, where $\alpha = (\lambda_1 + \lambda_2)/2$, $\beta = (\lambda_1 - \lambda_2)/2$, and

$$P = \frac{(A - \alpha I)^2 - \beta^2 I}{(\lambda_3 - \alpha)^2 - \beta^2}, \quad J = \frac{1}{\beta} [A - \alpha I - (\lambda_3 - \alpha)P].$$

Also, $PJ = JP = 0$, $P^2 = P$, $J^{2n} = I - P$, $n = 1, 2, \dots$, $J^{2n+1} = J$, $n = 0, 1, 2, \dots$. Consequently, for every $t \in \mathbb{R}$ we have

$$(7) \quad \begin{aligned} e^{tA} &= e^{\alpha t} [(\cosh \beta t)(I - P) + (\sinh \beta t)J] + e^{\lambda_3 t}P \stackrel{\text{or}}{=} \\ &\stackrel{\text{or}}{=} \frac{1}{2}e^{\lambda_1 t}(I + J - P) + \frac{1}{2}e^{\lambda_2 t}(I - J - P) + e^{\lambda_3 t}P. \end{aligned}$$

Proof: Case 1: We have $f_A(\lambda) = (\lambda - \lambda_0)^3$ and $N^3 = (A - \lambda_0 I)^3 = f_A(A) = 0$, by the Cayley-Hamilton theorem. Next, $e^{tA} = e^{t(\lambda_0 I + N)} = e^{t\lambda_0 I} e^{tN} = e^{\lambda_0 t} I (I + tN + (t^2/2)N^2 + 0 + \dots) = e^{\lambda_0 t} (I + tN + (t^2/2)N^2)$, as required.

Case 2: Denote $\lambda_3 - \lambda_0 = b$, $A - \lambda_0 I = B$ and observe that $P = \frac{1}{b^2}B^2$ and $N = B - bP = -\frac{1}{b}B(B - bI)$. Notice that $f_A(\lambda) = (\lambda - \lambda_0)^2(\lambda - \lambda_3)$ implying, by translation, $f_B(\lambda) = \lambda^2(\lambda - b)$. It follows $B^2(B - bI) = f_B(B) = 0$ (by the Cayley-Hamilton theorem), i.e. $B^3 = bB^2$ and $B^4 = b^2B^2$. After these preliminary evaluations we calculate $P^2 = B^4/b^4 = b^2B^2/b^4 = P$, as required. Next, we have $NP = PN = -\frac{1}{b^3}B^2B(B - bI) = -\frac{1}{b^3}Bf_B(B) = 0$, and $N^2 = -\frac{1}{b^2}B^2(B - bI)^2 = \frac{1}{b^2}f_B(B)(B - bI) = 0$, as required. Finally,

$$\begin{aligned} e^{tA} &= e^{t[\lambda_0 I + bP + N]} = e^{\lambda_0 t} I e^{btP} e^{tN} = \\ &= e^{\lambda_0 t} I \left(I + P \sum_{n=1}^{\infty} \frac{t^n b^n}{n!} \right) (I + tN) = e^{\lambda_0 t} [I - P + e^{bt}P](I + tN) = \\ &= e^{\lambda_0 t}(I - P + tN) + e^{\lambda_3 t}P, \end{aligned}$$

as required.

Case 3: We denote $A - \alpha I = B$ and $\lambda_3 - \alpha = b$ and observe that $P = \frac{1}{b^2 + \omega^2}(B^2 + \omega^2 I)$ and $J = \frac{1}{\omega}(B - bP)$. We have $f_A(\lambda) = [(\lambda - \alpha)^2 + \omega^2](\lambda - \lambda_3)$

which implies $f_B(\lambda) = (\lambda^2 + \omega^2)(\lambda - b)$. By the Cayley-Hamilton theorem, it follows $(B^2 + \omega^2 I)(B - bI) = 0$ implying also $B^3 + \omega^2 B = b(B^2 + \omega^2 I)$ and $B^4 + \omega^2 B^2 = b^2(B^2 + \omega^2 I)$. Next we calculate $PB = BP = \frac{1}{b^2 + \omega^2}(B^3 + \omega^2 B) = \frac{b}{b^2 + \omega^2}(B^2 + \omega^2 I) = bP$. It follows $B^2 P = b^2 P$ which helps us to show that P is a projection, i.e. $P^2 = \frac{1}{b^2 + \omega^2}(B^2 + \omega^2 I)P = \frac{1}{b^2 + \omega^2}(B^2 P + \omega^2 P) = \frac{1}{b^2 + \omega^2}(b^2 P + \omega^2 P) = P$ and also $PJ = JP = \frac{1}{\omega}(B - bP)P = \frac{1}{\omega}(bP - bP) = 0$. Finally, we have $J^2 = \frac{1}{\omega^2}(B - bP)^2 = \frac{1}{\omega^2}(B^2 - 2bBP + b^2 P) = \frac{1}{\omega^2}(B^2 - b^2 P) = \frac{1}{\omega^2}[(b^2 + \omega^2)P - \omega^2 I - b^2 P] = -(I - P)$, as required. The rest of the formulas for the powers of J follow immediately. Next, we calculate $e^{\omega t J}$:

$$\begin{aligned} e^{\omega t J} &= \sum_{n=0}^{\infty} \frac{(\omega t)^n}{n!} J^n = I + \sum_{n=1}^{\infty} \frac{(\omega t)^{2n}}{(2n)!} J^{2n} + \sum_{n=0}^{\infty} \frac{(\omega t)^{2n+1}}{(2n+1)!} J^{2n+1} = \\ &= P + (I - P) + \sum_{n=1}^{\infty} \frac{(-1)^n (\omega t)^{2n}}{(2n)!} (I - P) + \sum_{n=0}^{\infty} \frac{(-1)^n (\omega t)^{2n+1}}{(2n+1)!} J = \\ &= P + (\cos \omega t)(I - P) + (\sin \omega t)J. \end{aligned}$$

Finally, we calculate

$$\begin{aligned} e^{tA} &= e^{t[\alpha I + bP + \omega J]} = e^{\alpha t I} e^{bt P} e^{\omega t J} = e^{\alpha t} [(I - P) + e^{bt} P] \times \\ &\quad \times [P + (\cos \omega t)(I - P) + (\sin \omega t)J]. \end{aligned}$$

The last leads to formula (6) after standard manipulations.

Case 4: We denote, as before, $A - \alpha I = B$ and $\lambda_3 - \alpha = b$ and observe that $P = \frac{1}{b^2 - \beta^2}(B^2 - \beta^2 I)$ and $J = \frac{1}{\beta}(B - bP)$. Next, we have $f_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ which implies, by translation, $f_B(\lambda) = (\lambda^2 - \beta^2)(\lambda - b)$. It follows $(B^2 - \beta^2 I)(B - bI) = 0$, by the Cayley-Hamilton theorem. Thus $B^3 - \beta^2 B = b(B^2 - \beta^2 I)$ and $B^4 - \beta^2 B^2 = b^2(B^2 - \beta^2 I)$. Now the relations between P and J follow exactly as in Case 3. Next we calculate:

$$\begin{aligned} e^{tA} &= e^{t[\alpha I + bP + \beta J]} = e^{\alpha t I} e^{bt P} e^{\beta t J} = e^{\alpha t} [I - P + e^{bt} P] \times \\ &\quad \times \left[I + \sum_{n=1}^{\infty} \frac{(\beta t)^{2n}}{(2n)!} (I - P) + \sum_{n=0}^{\infty} \frac{(\beta t)^{2n+1}}{(2n+1)!} J \right] = [e^{\alpha t} (I - P) + e^{\lambda_3 t} P] \times \\ &\quad \times [P + (\cosh \beta t)(I - P) + (\sinh \beta t)J] = e^{\alpha t} [(\cosh \beta t)(I - P) + (\sinh \beta t)J] + \\ &\quad e^{\lambda_3 t} P, \end{aligned}$$

as required. \blacktriangle

The formulas (4)-(7) show that we do not need the eigenvectors of A in order to calculate e^{tA} . However, the eigenvectors of A as well as canonical forms and canonical bases (if needed for classification, graphing, etc.) can be easily extracted from the matrices N, P and J . The next corollary follows easily from the above theorem and we leave the proof to the reader.

Corollary 4.1 (Eigenvectors, Canonical Forms and Bases) *Under the notation of the previous theorem we have the following:*

Case 1: *If $N^2 \neq O$, then $E_{\lambda=\lambda_0}(A) = \text{Im}(N^2)$. In particular, either of the non-zero columns of N^2 is an eigenvector of A . We have $A = T \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{pmatrix} T^{-1}$, where $T = [N^2 \vec{x} | N\vec{x} | \vec{x}]$ and $\vec{x} \in \mathbb{R}^3$, $N^2 \vec{x} \neq \vec{0}$. If $N = O$, then $A = \lambda_0 I$. If $N^2 = O$ and $N \neq O$, then $E_{\lambda=\lambda_0}(A) \supsetneq \text{Im}(N)$. To complete a Jordan basis we can select an additional eigenvector by easy inspection of the columns of N .*

Case 2: *If $N \neq O$, then $E_{\lambda=\lambda_0}(A) = \text{Im}(N)$ and $E_{\lambda=\lambda_3}(A) = \text{Im}(P)$. In particular, either of the non-zero columns of N and P is an eigenvector of A . We have $A = T \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix} T^{-1}$, where*

$$T = [(A - \lambda_0 I)(A - \lambda_3 I)\vec{x} | (A - \lambda_3 I)\vec{x} | P\vec{y}],$$

and $\vec{x} \in \mathbb{R}^3$, $(A - \lambda_0 I)(A - \lambda_3 I)\vec{x} \neq \vec{0}$ and $P\vec{y} \neq \vec{0}$. If $N = O$, then $A = \text{diag}(\lambda_0, \lambda_0, \lambda_3)$.

Case 3: *The only eigenspace in \mathbb{R}^3 is $E_{\lambda=\lambda_3} = \text{Im}(P)$. Here we have $A = T \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} T^{-1}$, where $T = [J^2 \vec{x} | J\vec{x} | P\vec{y}]$ and $\vec{x}, \vec{y} \in \mathbb{R}^3$, $\vec{x} \neq \vec{0}$,*

$P\vec{y} \neq \vec{0}$. Alternatively, in \mathbb{C}^3 the matrix A is diagonalizable with eigenvalues $\alpha \pm i\omega$, λ_3 and eigenvectors $(J \mp iJ^2)\vec{x}$, $\vec{x} \neq \vec{0}$, $P\vec{y} \neq \vec{0}$, respectively.

Case 4: *We have $E_{\lambda=\lambda_{1,2}}(A) = \text{Im}(J \pm J^2)$, respectively, and $E_{\lambda=\lambda_3}(A) = \text{Im}(P)$. Thus the matrix A is diagonalizable in the basis consisting of the eigenvectors listed above.*

Here are several examples.

Example 4.1 *Let $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$, be a system*

of three first order homogeneous ordinary differential equations with constant coefficients with initial conditions $\vec{x}(\vec{0}) = (1, 0, 2)$. We apply the formula for the solution $\vec{x}(t) = e^{tA} \vec{x}_0$. It remains to calculate e^{tA} . We leave to the reader to check that $\lambda_1 = \lambda_2 = \lambda_3 = -1$ (Case 1 in Theorem 4.1). We calculate

$$N = A + I = \begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}, \quad N^3 = O.$$

The formula (4) gives

$$e^{tA} = e^{\lambda_0 t} \left(I + tN + \frac{t^2}{2} N^2 \right) = e^{-t} \begin{pmatrix} 1 + 3t + t^2, & -t - t^2/2, & 2t + t^2/2 \\ 5t + t^2, & 1 - 2t - t^2/2, & 3t + t^2/2 \\ -t - t^2, & t^2/2, & 1 - t - t^2/2 \end{pmatrix}.$$

$$\text{The solution of the system is } \vec{x}(t) = e^{tA} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = e^{-t} \begin{pmatrix} 1 + 7t + 2t^2 \\ 11t + 2t^2 \\ 2 - 3t - 2t^2 \end{pmatrix}.$$

Remark 4.1 (Jordan Basis and Eigenvectors) We do not need a Jordan basis (and the eigenvectors) of A to calculate e^{tA} . However, a Jordan basis for A , if needed, can be easily extracted from the above calculations with the help of Corollary 4.1. For example, $\{N^2 \vec{e}_3, N \vec{e}_3, \vec{e}_3\}$ (where \vec{e}_3 is the third vector in the standard basis in \mathbb{R}^3), forms a Jordan basis for A . Also, we have $E_{\lambda=-1} = \text{Im}(N^2)$. In particular, either of the columns of N^2 is an eigenvector of A for $\lambda = -1$. We leave to the reader to check that $A = TCT^{-1}$, where $C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $T = [N^2 \vec{e}_3 \mid N \vec{e}_3 \mid \vec{e}_3] = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix}$.

The columns of T forms a Jordan basis for A .

Example 4.2 Let $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$, be a system of

three first order homogeneous ordinary differential equations with constant coefficients with initial conditions $\vec{x}(\vec{0}) = (1, 0, 1)$. We leave to the reader to check that $\lambda_1 = \lambda_2 = -1 = \lambda_0$ and $\lambda_3 = 3$ (Case 2 in Theorem 4.1). We

calculate

$$A + I = \begin{pmatrix} 2 & -3 & 4 \\ 4 & -6 & 8 \\ 6 & -7 & 8 \end{pmatrix}, \quad P = \left(\frac{A + I}{4} \right)^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{pmatrix},$$

$$N = A + I - 4P = \begin{pmatrix} 2 & -3 & 4 \\ 4 & -6 & 8 \\ 6 & -7 & 8 \end{pmatrix} - 4 \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}.$$

Notice that $N^2 = O$. The formula (5) gives

$$e^{tA} = e^{-t} \begin{pmatrix} 1 - 2t & t & 0 \\ -4t & 1 + 2t & 0 \\ -2t & t & 1 \end{pmatrix} + (e^{3t} - e^{-t}) \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -2te^{-t} + e^{3t}, & (1+t)e^{-t} - e^{3t}, & -e^{-t} + e^{3t} \\ (-2 - 4t)e^{-t} + 2e^{3t}, & (3 + 2t)e^{-t} - 2e^{3t}, & -2e^{-t} + 2e^{3t} \\ -4te^{-t} + 2e^{3t}, & (2 + t)e^{-t} - 2e^{3t}, & -2e^{-t} + 2e^{3t} \end{pmatrix}.$$

$$For the solution we have \vec{x}(t) = e^{tA} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (-1 - 2t)e^{-t} + 2e^{3t} \\ (-4 - 4t)e^{-t} + 4e^{3t} \\ (-2 - 4t)e^{-t} + 4e^{3t} \end{pmatrix}.$$

Remark 4.2 (Jordan Basis and Eigenvectors) We just calculated e^{tA} without the Jordan basis and eigenvectors of A . They, however (if needed), can be easily extracted from the above calculations with the help of Corollary 4.1. For example, $E_{\lambda=-1} = \text{Im}(N)$ and $E_{\lambda=3} = \text{Im}(P)$. In particular, the first and second columns of N are eigenvectors of A for $\lambda = -1$ and either of the columns of P is an eigenvector of A and $\lambda = 3$. We leave

to the reader to check that $A = TCT^{-1}$, where $C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and

$$T = [(A - \lambda_0 I)(A - \lambda_3 I)\vec{x} | (A - \lambda_3 I)\vec{x} | P\vec{y}] = \begin{pmatrix} 8 & -1 & 1 \\ 16 & 2 & 2 \\ 8 & 3 & 2 \end{pmatrix} \text{ for } \vec{x} = \vec{y} = \vec{e}_1.$$

The columns of T forms a Jordan basis for A .

Example 4.3 Let $\vec{x}'(t) = A\vec{x}(t)$, where $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, be a system of three first order homogeneous ordinary differential equations with constant

coefficients with initial conditions $\vec{x}(0) = \vec{e}_2$, where \vec{e}_2 is the second vector in the standard basis in \mathbb{R}^3 . We have $\lambda_{1,2} = 1 \pm i$, i.e. $\gamma = \omega = 1$, and $\lambda_3 = 3$ (Case 3 in Theorem 4.1). We calculate

$$P = \frac{(A - \gamma I)^2 + \omega^2 I}{(\lambda_3 - \gamma)^2 + \omega^2} = \frac{1}{5} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$J = \frac{1}{\omega} [A - \gamma I - (\lambda_3 - \gamma)P] = \frac{1}{5} \begin{pmatrix} 0 & 1 & -5 \\ 0 & 0 & 0 \\ 5 & -2 & 0 \end{pmatrix},$$

$$\begin{aligned} e^{tA} &= (e^{\lambda_3 t} - e^{\gamma t} \cos \omega t)P + e^{\gamma t}[(\cos \omega t)I + (\sin \omega t)J] = \\ &= \frac{1}{5} \begin{pmatrix} 5e^t \cos t, & 2e^{3t} - e^t(2 \cos t - \sin t), & -5e^t \sin t \\ 0, & 5e^{3t}, & 0 \\ 5e^t \sin t, & e^{3t} - e^t(\cos t + 2 \sin t), & 5e^t \cos t \end{pmatrix}. \end{aligned}$$

$$For the solution we have $\vec{x}(t) = e^{tA} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2e^{3t} - e^t(2 \cos t - \sin t) \\ 5e^{3t} \\ e^{3t} - e^t(\cos t + 2 \sin t) \end{pmatrix}.$$$

Remark 4.3 (Canonical Forms and Eigenvectors) As before, we just calculated e^{tA} without the Jordan basis and eigenvectors of A . They, however (if needed), can be easily extracted from the above calculations with the help of Corollary 4.1. In this case $E_{\lambda=\lambda_3} = \text{Im}(P)$ and we have $A = TCT^{-1}$, where $C = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $T = [J^2 \vec{x} \mid J\vec{x} \mid P\vec{y}] = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix}$ for $\vec{x} = \vec{e}_1$ and $\vec{y} = 5\vec{e}_2$. The columns of T form a canonical basis for A .

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